KVL Example

Let's start by applying Kirchoff's Voltage Law:

$$\underbrace{V_s}_{\text{Voltage gain}} - \underbrace{V_R}_{\text{voltage drop}} - \underbrace{V_c}_{\text{Voltage drop}} = 0 \qquad (12)$$

By Ohm's Law:

$$V_R = i\,R \qquad (13)$$

Combining equations (1) and (2):

$$V_s - iR - V_c = 0 \qquad (14)$$

Let the following equation apply for capacitors:

$$i = C\,\frac{dV_c}{dt} \qquad (15)$$

We can combine equations (3) and (4) since the same current runs through both elements.

$$V_s - RC\,\frac{dV_c}{dt} - V_c = 0 \qquad (16)$$

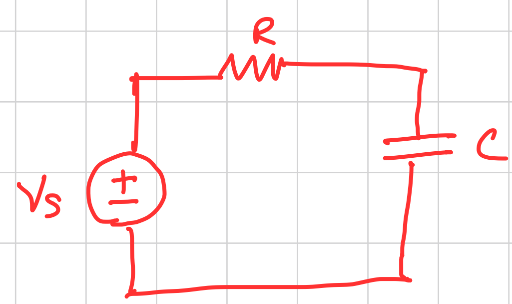
Isolating V_c to one side and dividing thru by C:

$$\frac{dV_c}{dt} + \frac{1}{RC}V_c = \frac{V_s}{RC} \qquad (17)$$

KCL Example:

$$i_0 = i_1 + i_2 \qquad (18)$$

Series RC circuit Derivation:



We already derived the characteristic differential equation for a series RC circuit. Repeating eq (17):

$$\frac{dV_c}{dt} + \frac{1}{RC} V_c = \frac{V_s}{RC}$$

To solve this equation, let's use the integrating factor (IF) method. First, let's define the integrating factor as:

$$IF \equiv \exp\left(\int \frac{1}{RC}\, dt\right)$$

$$IF = \exp\left(\frac{1}{RC} \int dt\right)$$

$$IF = \exp\left(\frac{1}{RC} t\right)$$

* Note: There technically should be a constant of integration, but we will do some more integrating later, and the constants will merge. Thus, it is not necessary to write it out in this step.

Now, let's multiply both sides of eq (17) by the integrating factor.

$$\exp\left(\frac{1}{RC}t\right)\left[\frac{dV_c}{dt} + \frac{1}{RC}V_c\right] = \exp\left(\frac{1}{RC}t\right)\frac{V_s}{RC} \tag{19}$$

By the inverse product and chain rule, we can rewrite the left side of the equation:

$$\left[\frac{d}{dt}\right]\left[V_c \exp\left(\frac{1}{RC}t\right)\right] = \exp\left(\frac{1}{RC}t\right)\frac{V_s}{RC}$$

Moving the dt term over to the right and integrating:

$$\int d\left[V_c \exp\left(\frac{1}{RC}t\right)\right] = \int \left[\exp\left(\frac{1}{RC}t\right)\frac{V_s}{RC}\right] dt$$

$$V_c \exp\left(\frac{1}{RC}t\right) = \frac{V_s}{RC}\exp\left(\frac{1}{RC}t\right)\cdot\frac{1}{\frac{1}{RC}} + \underbrace{A}_{\text{const. of integration}}$$

$$V_c \exp\left(\frac{1}{RC}t\right) = V_s \exp\left(\frac{1}{RC}t\right) + A \tag{20}$$

Solving for V_c:

$$V_c = A\exp\left(-\frac{1}{RC}t\right) + V_s \tag{21}$$

Now, we need to apply our initial condition. At $t=0$, let $V_c = V_{c,0}$. Substituting this into equation (21) to solve for A:

$$V_{c,0} = A\exp(0) + V_s$$

$$A = V_{c,0} - V_s \tag{22}$$

Combining eq. (21) and (22):

$$V_C = \left(V_{C,0} - V_S\right) \exp\left[-\frac{1}{RC} t\right] + V_S \qquad (23).$$

Equation (23) is a rather rigorous solution to an RC circuit. In most applications, we judiciously choose $V_{C,0}$ to be 0. Thus, equation (23) would simplify to:

$$V_C = V_S \left[1 - \exp\left(-\frac{1}{RC} t\right)\right] \qquad (24)$$

To graph this:

- when $t=0$

$$V_c(t=0) = V_S\left[1 - \exp(0)\right]$$

$$V_c(t=0) = V_S(1-1)$$

$$V_c(t=0) = 0$$

- When $t \to \infty$:

$$V_c(t\to\infty) = V_S\left[1 - \exp(-\infty)\right]$$

$$V_c(t\to\infty) = V_S\left[1 - \frac{1}{\exp(\infty)}\right]$$

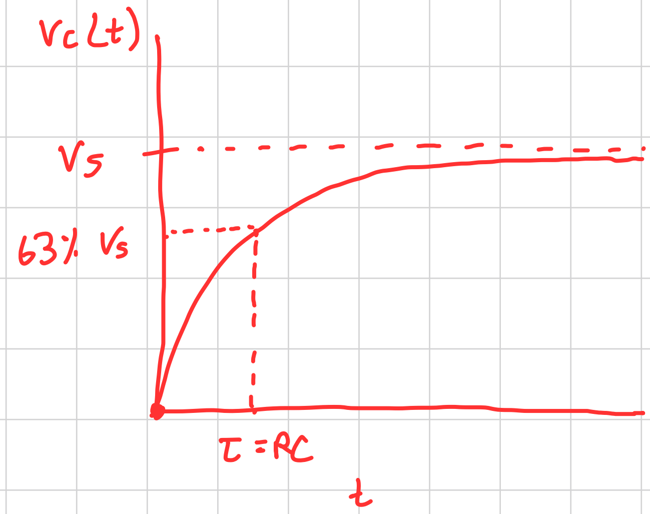
$$V_c(t\to\infty) = V_S$$

- When $t = RC$
  - ↳ this is referred to as the time constant:

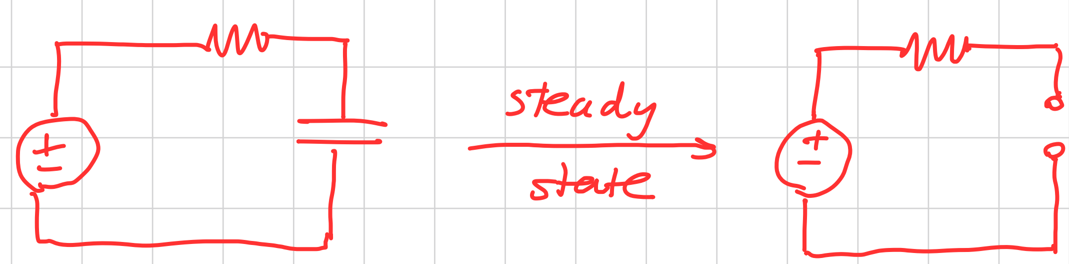
$$V_c(t=RC) = V_S\left[1 - \exp\left(-\frac{RC}{RC}\right)\right]$$

$$V_c(t=RC) = V_S\left[1 - \exp(-1)\right]$$

$$V_c(t=RC) \approx 0.63\, V_S \qquad \text{“63\% completed”}$$

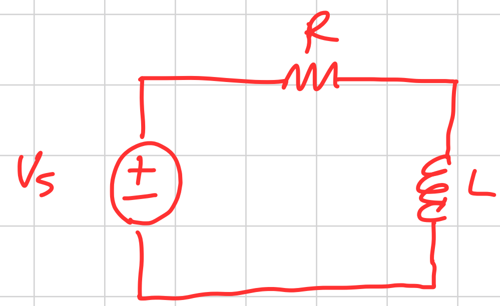


Think about this graph. As $t \to \infty$ "steady state", the voltage difference is the same as the source. Effectively, the capacitor acts as an open circuit as $t \to \infty$ ($\sim$4-5 time constants):



No current can pass through at steady state (for a DC source).

Series RL Circuit Derivation :



Applying Kirchoff's Voltage Law:

$$V_R + V_L = V_s \tag{25}$$

By Ohm's Law:

$$V_R \equiv iR \tag{26}$$

The characteristic equation for an inductor is given by:

$$V_L = L\frac{di}{dt} \tag{27}$$

Combining equations (25)-(27):

$$iR + L\frac{di}{dt} = V_s \tag{28}$$

Dividing through by L:

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V_s}{L} \tag{29}$$

Equation (29) is the characteristic differential eq. for an RL circuit. Let the integrating factor be defined as:

$$IF = \exp\left(\int \frac{R}{L}\,dt\right)$$

$$IF = \exp\left(\frac{R}{L}t\right) \tag{30}$$

Multiplying both sides of equation (5) by the integrating factor:

$$\exp\left(\frac{R}{L}t\right)\left[\frac{di}{dt}+\frac{R}{L}i\right]=\exp\left(\frac{R}{L}t\right)\frac{V_s}{L} \qquad (31)$$

By the inverse product and chain rule, the left side of the equation becomes:

$$\left(\frac{d}{dt}\right)\left[i\exp\left(\frac{R}{L}t\right)\right]=\exp\left(\frac{R}{L}t\right)\frac{V_s}{L} \qquad (32)$$

Moving the dt term over and integrating:

$$\int d\left[i\exp\left(\frac{R}{L}t\right)\right]=\int\left[\exp\left(\frac{R}{L}t\right)\frac{V_s}{L}\right]dt$$

$$i\exp\left(\frac{R}{L}t\right)=\frac{V_s}{L}\exp\left(\frac{R}{L}t\right)\cdot\frac{1}{\frac{R}{L}}+A \qquad (33)$$

Dividing thru by $\exp\left(\frac{R}{L}t\right)$:

$$i=\frac{V_s}{R}+A\exp\left(-\frac{R}{L}t\right) \qquad (34)$$

To solve for A, let $i(t=0)=i_0$:

$$i_0=\frac{V_s}{R}+A\exp(0)$$

$$A=i_0-\frac{V_s}{R} \qquad (35)$$

Combining equations (34) and (35):

$$i=\left(i_0-\frac{V_s}{R}\right)\exp\left(-\frac{R}{L}t\right)+\frac{V_s}{R} \qquad (36)$$

Again, equation (36) may be too rigorous for practical applications.
Let $i_0 = 0$:

$$i = \frac{-V_s}{R} \exp\left(\frac{-R}{L} t\right) + \frac{V_s}{R}$$

$$i = \frac{V_s}{R}\left[1 - \exp\left(\frac{-R}{L} t\right)\right] \qquad (37)$$

Let's graph this to understand the function of inductors:

- At $t = 0$:

$$i = \frac{V_s}{R}\left[1 - \exp(0)\right]$$

$$i = \frac{V_s}{R}\left[1 - 1\right]$$

$$i = 0$$

- At $t \to \infty$

$$i = \frac{V_s}{R}\left[1 - \exp(-\infty)\right]$$

$$i = \frac{V_s}{R}\left[1 - \frac{1}{\exp(\infty)}\right]$$

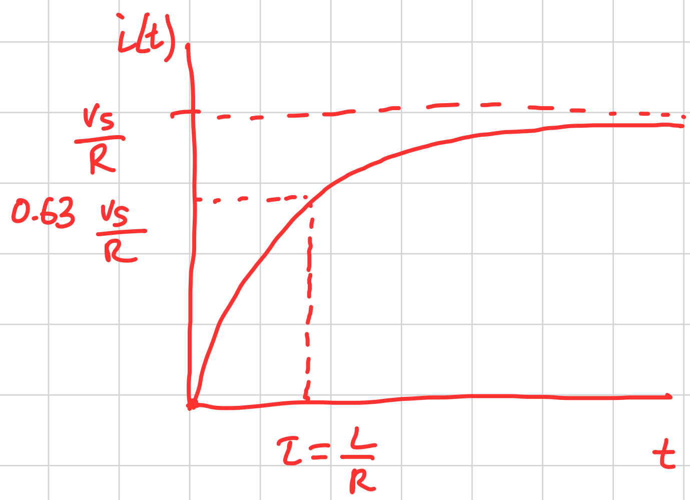
$$i = \frac{V_s}{R}$$

- At $t = \frac{L}{R} = \tau$:

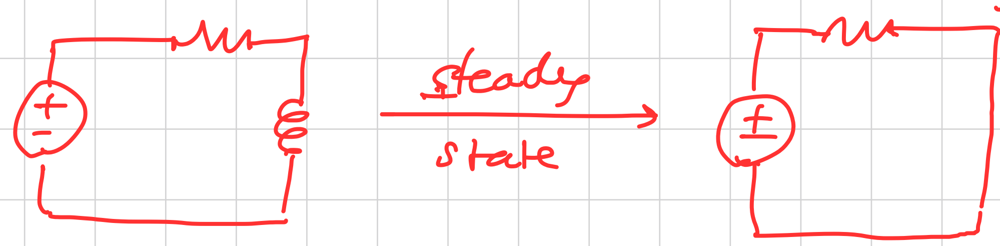
$$i = \frac{V_s}{R}\left[1 - \exp\left(\frac{-R}{L} \cdot \frac{L}{R}\right)\right]$$

$$i = \frac{V_s}{R}\left[1 - \exp(-1)\right]$$

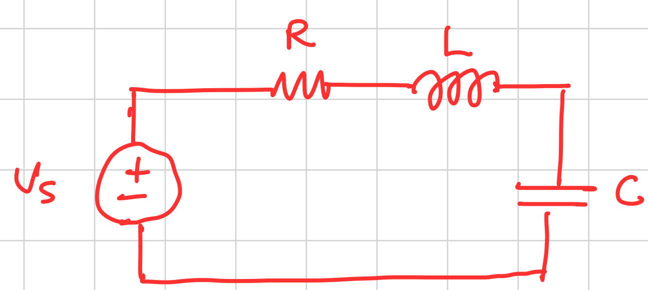
$$i \approx 0.63 \frac{V_s}{R}$$ "63% completed"



The current reaches its maximum value as $t \to \infty$. This describes inductors as a short circuit, since there is no resistance at steady state.



Series RLC Circuits



Applying Kirchoff's Voltage Law:

$$V_R + V_L + V_C = V_S \qquad (38)$$

Ideally, we want to get equation (39) in terms of i. Let's start with equation (6) from the slides:

$$i = C\frac{dV_C}{dt}$$

We need to solve for V_C so that we can substitute this into eq (38):

$$\frac{dV_C}{dt} = \frac{1}{C}\, i(t)$$

Moving dt to the right side and integrating:

$$\int dv_c = \frac{1}{C}\int_0^t i(t)\,dt$$

$$V_c = \frac{1}{C}\int_0^t i(t)\,dt \qquad (39)$$

Combining eqns (1), (9), (38), and (39):

$$iR + L\frac{di}{dt} + \frac{1}{C}\int_0^t i(t)\,dt = V_s \qquad (40)$$

Differentiating all of eq (40) with respect to t:

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = V_s \qquad (41)$$

Dividing through by L:

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = \frac{V_s}{L} \qquad (42)$$

Equation (42) represents the characteristic differential equation for a series RLC circuit. To solve for the homogeneous part, we set the left side equal to 0:

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0 \qquad (43)$$

Let's define a differential operator:

$$\gamma \equiv \frac{d}{dt}$$

Applying this to eq (43):

$$\gamma^2 i + \frac{R}{L}\gamma i + \frac{1}{LC}i = 0$$

$$i\left[\gamma^2+\frac{R}{L}\gamma+\frac{1}{LC}\right]=0$$

$$\gamma^2+\frac{R}{L}\gamma+\frac{1}{LC}=0 \qquad (44)$$

Solving for γ:

$$\gamma=\frac{-\frac{R}{L}\pm\sqrt{\left(\frac{R}{L}\right)^2-\frac{4}{LC}}}{2} \qquad (45)$$

The solution depends on the relative signs and magnitudes of L and R. We define three cases:

CASE 1: If γ_1 and γ_2 are real $\Rightarrow$ overdamped

$$i_h = \underbrace{K_1}_{\text{constants of integration}} \exp(\gamma_1 t) + K_2 \exp(\gamma_2 t) \qquad (46a)$$

CASE 2: If $\gamma_1=\gamma_2=\gamma_0$ $\Rightarrow$ critically damped.

$$i_h = K_1 \exp(\gamma_0 t) + K_2 t \exp(\gamma_0 t) \qquad (46b)$$

CASE 3: If γ_1 and γ_2 are complex $\Rightarrow$ underdamped

$\left(\gamma_{1,2} = \mu + j\nu \right.$ use j instead of i to avoid confusion w/ current $\left. j \equiv \sqrt{-1} \right)$

$$i_h = \exp(\mu t)\left[K_1\cos(\nu t)+K_2\sin(\nu t)\right] \qquad (46c)$$

Equations (46a)–(46c) just describe the <u>homogenous</u> solution. What about the particular solution?

The particular solution can be found by taking the steady state value:

$$i_p \equiv \lim_{t \to \infty} i(t) \qquad (47)$$

This can usually be found by <u>inspection</u>

The full solution is given by the sum of the homogeneous and particular solutions:

$$i = i_h + i_p \qquad (48)$$